Convex polytopes, Boolean bases and linear homology

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1 Preliminaries

Definition 1. *$\mathcal{F}_n$ is the vector space spanned by the flag vectors of all n-dimensional convex polytopes, and $\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{F}_n$.*

Clearly, $\dim \mathcal{F}_0 = 1$, being generated by the flag vector of a point.

Definition 2. *C is the* **cone** *(or pyramid) linear operator $C : \mathcal{F}_n \to \mathcal{F}_{n+1}$.*

I is the **cylinder** *(or prism) linear operator $I : \mathcal{F}_n \to \mathcal{F}_{n+1}$.*

P is the **product** *bilinear operator $P : \mathcal{F}_n \times \mathcal{F}_m \to \mathcal{F}_{n+m}$.*

The operators C, I, and P are induced by the cone, cylinder and Cartesian product operators respectively on convex polytopes.

Definition 3. *We write $D : \mathcal{F}_n \to \mathcal{F}_{n+2}$ for the linear operator $D = IC - CC$.*

Throughout w will denote a finite word in C and D. We denote by $\dot{w}$ the result of applying w to (the flag vector of) a point.

Theorem 4 (Fine. The IC equation)**.** *As operators on $\mathcal{F}$, we have $ID = DI$.*

Theorem 5 (Bayer and Billera)**.** *The vectors $\dot{w}$ provide a basis for $\mathcal{F}$.*

Thus $\dim \mathcal{F}_n = F_{n+1}$, the $(n+1)$-st Fibonacci number.

2 Structure constants By Bayer-Billera, the $\dot{w}$ provide a basis for $\mathcal{F}$. The linear operators C and D are easy to write down in this basis. We have $C\dot{w} = \dot{x}$ where $x = Cw$, and similarly for D. However, the rule for P is complicated.

Throughout $e = \{e_i\}$ will be a basis for $\mathcal{F}$. As with the $\dot{w}$, we insist that each e_i lies in some $\mathcal{F}_n$. For each basis e the operators C, D and P determine and are determined by structure constants.

Definition 6. *The equation $g_i^e(e_j) = \delta_{ij}$ defines the* **dual basis** *to e. It is a basis for the linear functions on $\mathcal{F}$. (Here δ_{ij} is the Kronecker delta.)*

Definition 7. *The* **structure constants** *for $\mathcal{F}$ in the basis e are:*

$$\lambda^e_{iw} = g^e_i(\dot{w}) \tag{1}$$
$$C^e_{ij} = g^e_i(Ce_j) \tag{2}$$
$$D^e_{ij} = g^e_i(De_j) \tag{3}$$
$$P^e_{ijk} = g^e_i(P(e_j, e_k)) \tag{4}$$

We may write C_{ij} for C^e_{ij} and so on if the basis e is understood.

3 Conjectures We have already noted that for the $\dot{w}$ basis all of the structure constants are particularly simple, except for P^e_{ijk}, which is complicated.

Definition 8. *We say that a basis b for $\mathcal{F}$ is* **Boolean** *if zero and one are the only values taken by the structure constants λ^b_{iw}, C^b_{ij} and so on (in the basis b).*

Conjecture 9. *There is exactly one Boolean basis for $\mathcal{F}$.*

Conjecture 10. *If X is a convex polytope with flag vector $v = f(X)$ then $g^b_i(v) \geq 0$, where b is as Conjecture 9.*

Conjecture 11. *There is a* **linear homology** *theory, whose Betti numbers are given by $g^b_i(X)$.*

By definition, each Betti number is the dimension of the corresponding homology space. Because the dimension of a vector space is non-negatice, it is trivial that Conjecture 11 implies Conjecture 10. This may be the only way to prove Conjecture 10.

4 Remarks Rarely will a C, D, P structure have a Boolean basis. This is because the structure has many more parameters than the basis e has. The C, D, P structure we have here comes from the geometry of convex polytopes.

Suppose linear homology exists. Then there are geometric reasons (which lie outside the scope of this note) to expect the e_i associated with its g_i to provide a Boolean basis for $\mathcal{F}$. Therefore, the main conjecture in this note is:

Conjecture 12. *There is a linear homology theory for convex polytopes, whose Betti numbers g_i provide a basis for the linear functions on $\mathcal{F}$.*

Granted this conjecture and the previously mentioned geometric reasons, the remaining conjectures are unsurprising. (If there are multiple Boolean bases, an additional condition should be found that selects the right one.)

The author has found a basis e for $\mathcal{F}$, which is Boolean for $n \leq 5$ and Boolean with a small number of exceptions for $n \leq 8$. The basis for $n = 5$ has $g_i(X) \geq 0$ for a very large collection of test polytopes (the zero-one polytopes and their polars). Indeed, the g_i for $n = 5$ are sharp for $g_i(X) \geq 0$ on these test polytopes.

This provides computational evidence for Conjecture 9. The author hopes, when time permits, to extend the Boolean basis to say $n \leq 15$. He expects Conjecture 9 to have a geometric proof, similar to his proof of the IC equation. Computation may reveal the geometric relations underlying such a proof.