Determining Subgrid Error in Numerical Modelling of Porous Media Flows

Mike Christie
Heriot-Watt University
Edinburgh, UK

Uncertainty Quantification

Contents

- Introduction
- Bayesian Framework
- Solution Error Models
- Applications
  - Porous media example
  - Gas Dynamics Example
- Conclusions

Uncertainty in Reservoir Description

How do we map between wells?

Uncertainty in Reservoir Description

Mathematics of Flow in Porous Media

- Conservation of Mass
- Conservation of Momentum
  - replaced by Darcy's law

$$\mathbf{v} = -\frac{k(\mathbf{x})}{\mu} \nabla p$$

- Conservation of Energy
  - most processes isothermal
- Equation of State

Equations governing flow

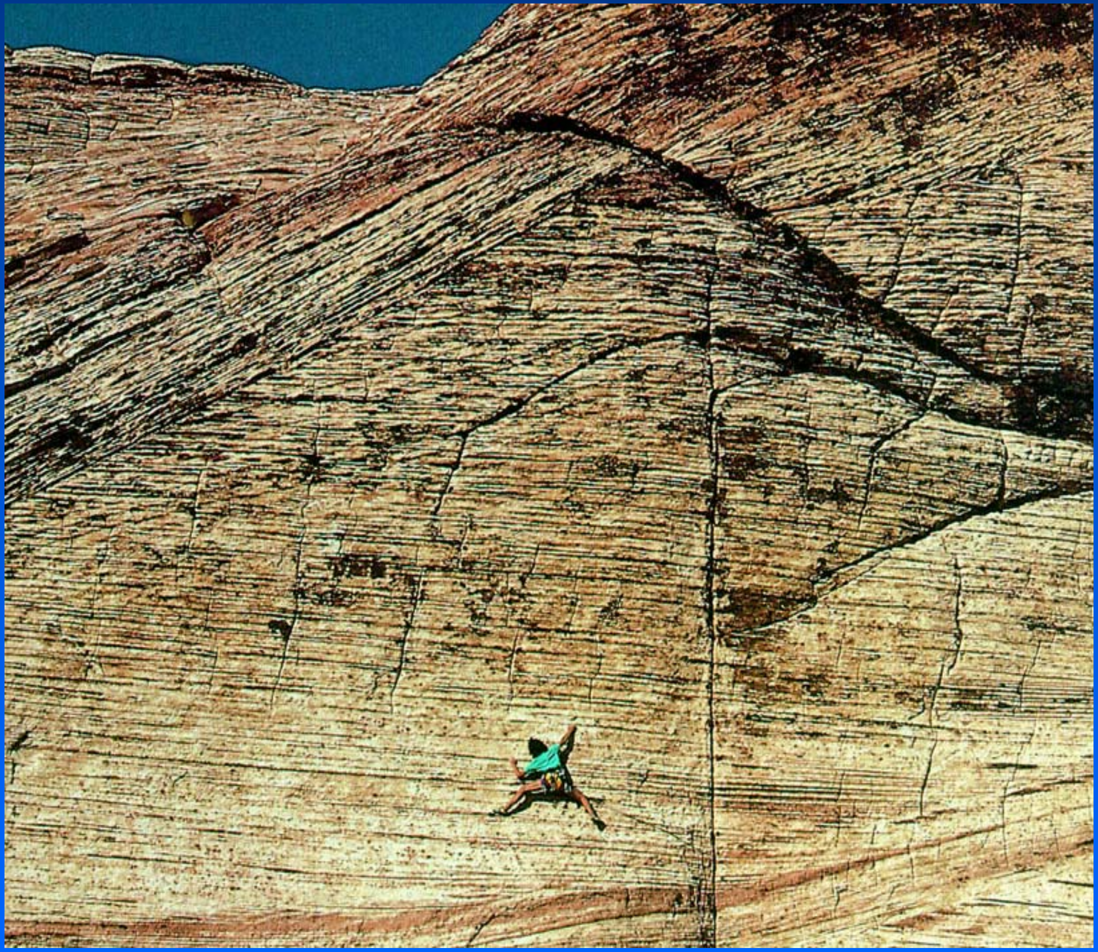
- Parabolic equation for pressure

$$c\frac{\partial p}{\partial t} = \nabla.\left(k(\mathbf{x})\left(\frac{k_{ro}(S)}{\mu_o} + \frac{k_{rw}(S)}{\mu_w} \right)\nabla p \right)$$

- Hyperbolic equation for saturation

$$\phi(\mathbf{x})\frac{\partial\left(\rho_o x_i S_o + \rho_g y_i S_g\right)}{\partial t} + \nabla.\left(\rho_o x_i \mathbf{v}_o + \rho_g y_i \mathbf{v}_g\right) = 0$$

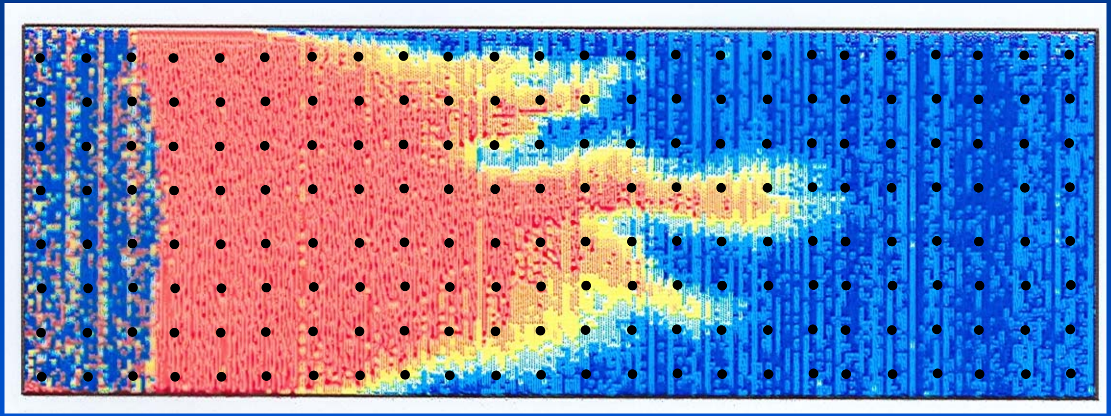
Data Collection



$$\phi(\mathbf{x}) = ?$$

$$k(\mathbf{x}) = ?$$

CT Scanned Rock Slab Experiment



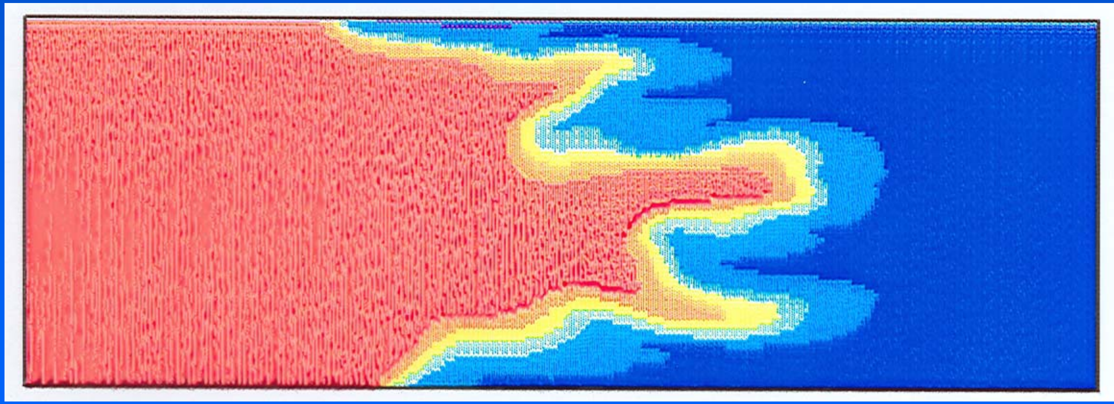
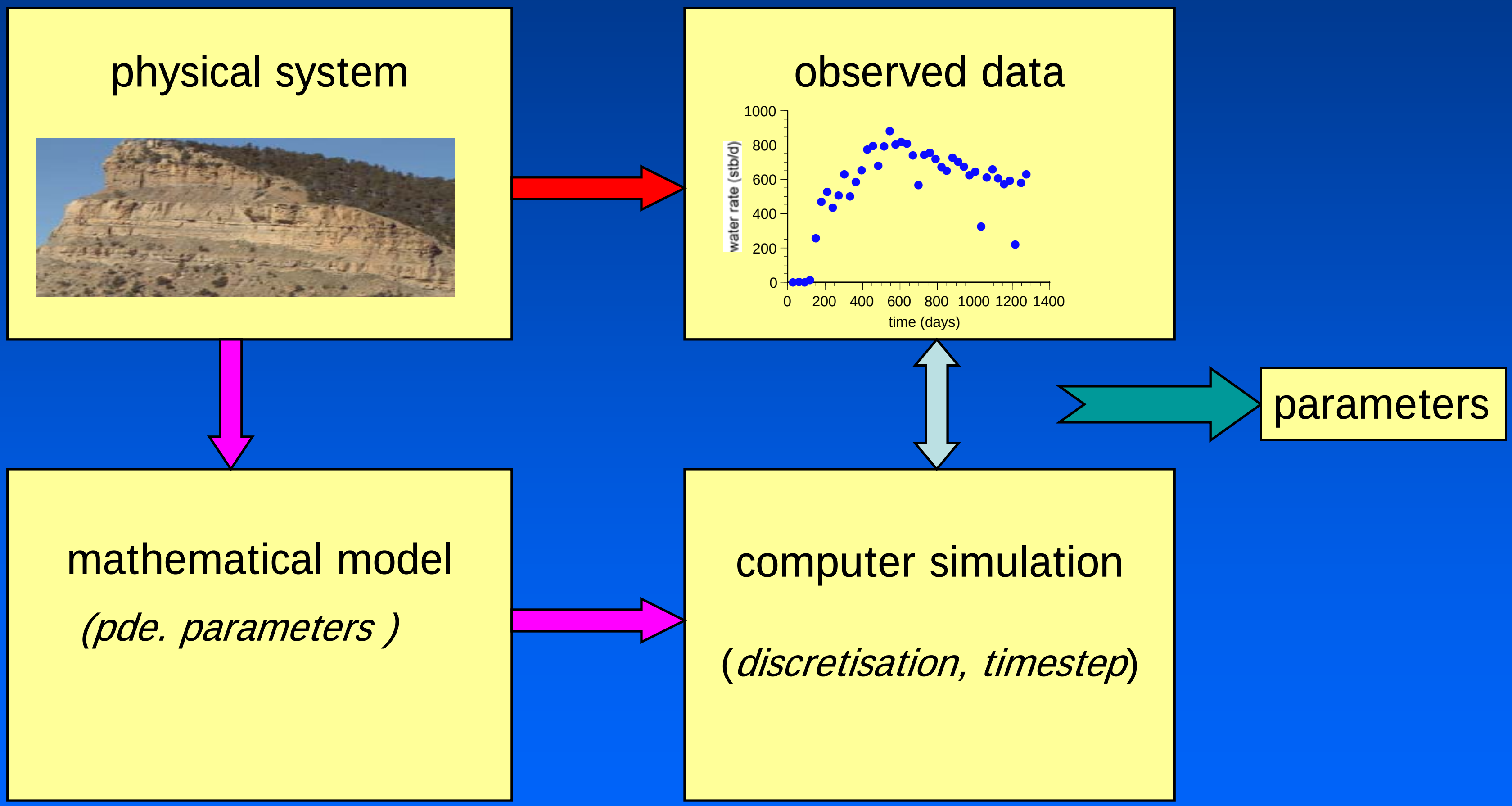
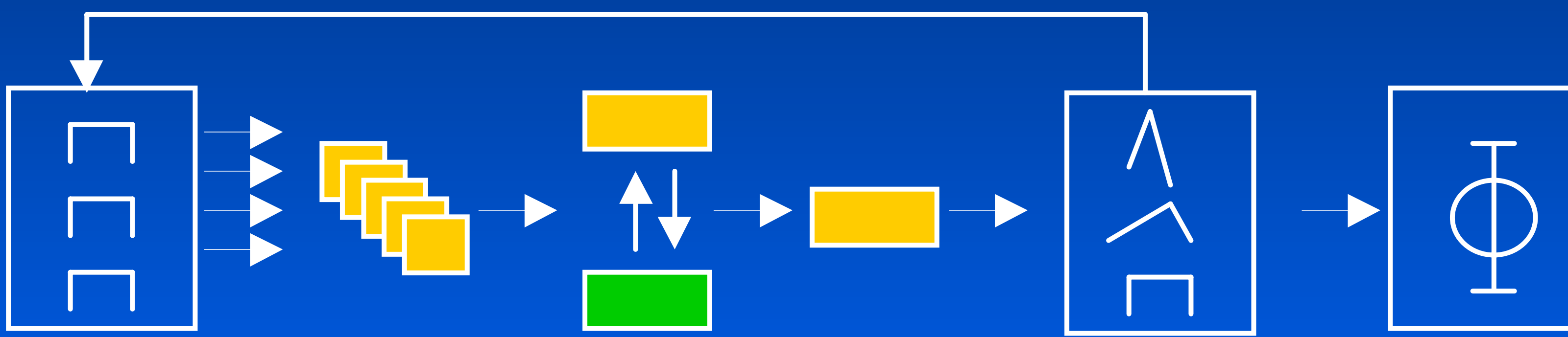


Image from Davies, Muggeridge, Jones, "Miscible Displacements in a Heterogeneous Rock: Detailed Measurements and Accurate Predictive Simulation" SPE 22615 (1991)

Reservoir Model Inverse Problem



Framework for History Matching



Sample from distribution

Generate multiple models

Compare production data & reservoir model

Evaluate misfit

Update probabilities using Bayes rule:

$$p(m\,|\,O) = \frac{p(O|m)p(m)}{p(O)}$$

Forecast with uncertainty

How to Determine Likelihood

- Difference between observation and model

$$obs - sim = (obs - true) - (sim - true)$$

$$= err_{obs} - err_{sim}$$

- Likelihood is the probability that the observed error and the model/simulation error add (subtract) to give discrepancy = 0

Likelihood Formula

- Likelihood given by convolution:

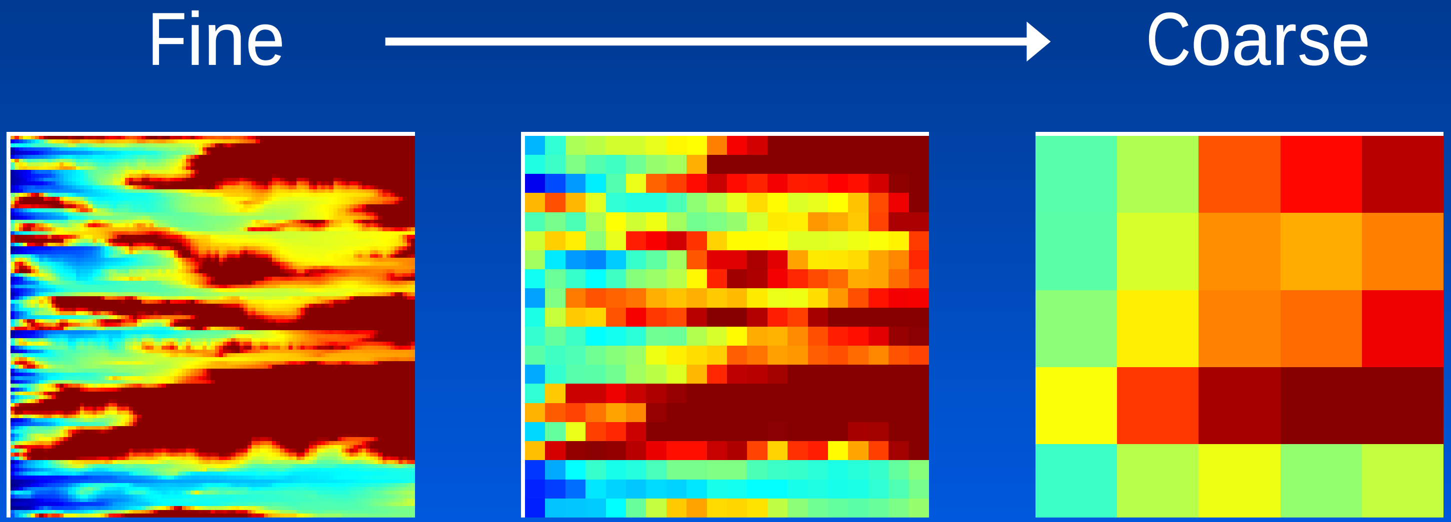
$$p(O\,|\,m) = \int p_{obs\_err}(x)\, p_{sim\_err}(x+y)\, dx$$

  - plus sign because errors are subtracted
- If errors are Gaussian
  - add covariance matrices
  - Note model error not necessarily zero mean

$$-\log p(O\,|\,m) = (\mathbf{o} - \mathbf{s} + \overline{\mathbf{e}})^T C^{-1} (\mathbf{o} - \mathbf{s} + \overline{\mathbf{e}})$$

$$C = C_d + C_m$$

Cell Size and Model Error

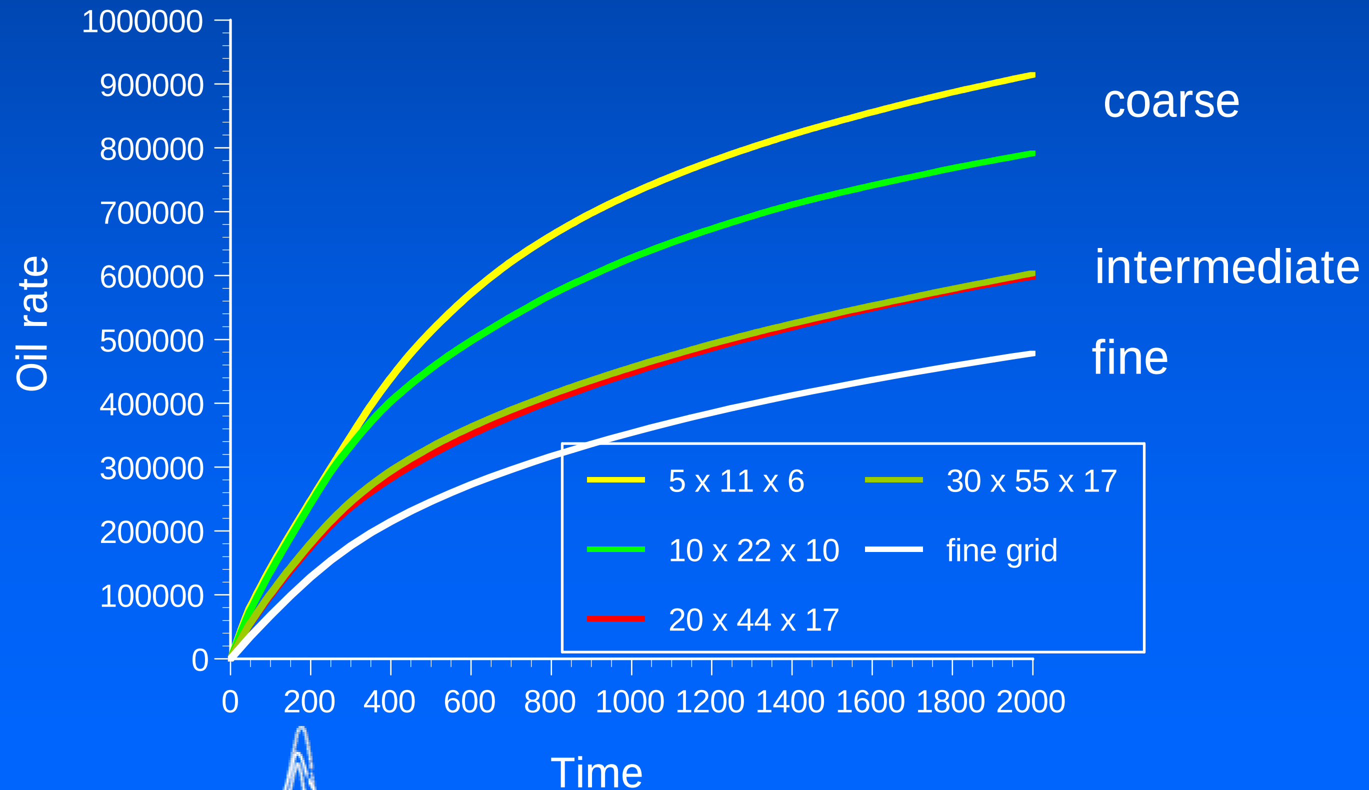


From fine to coarse:

- Reduced information
- Increased cell size

Solution Error Modelling

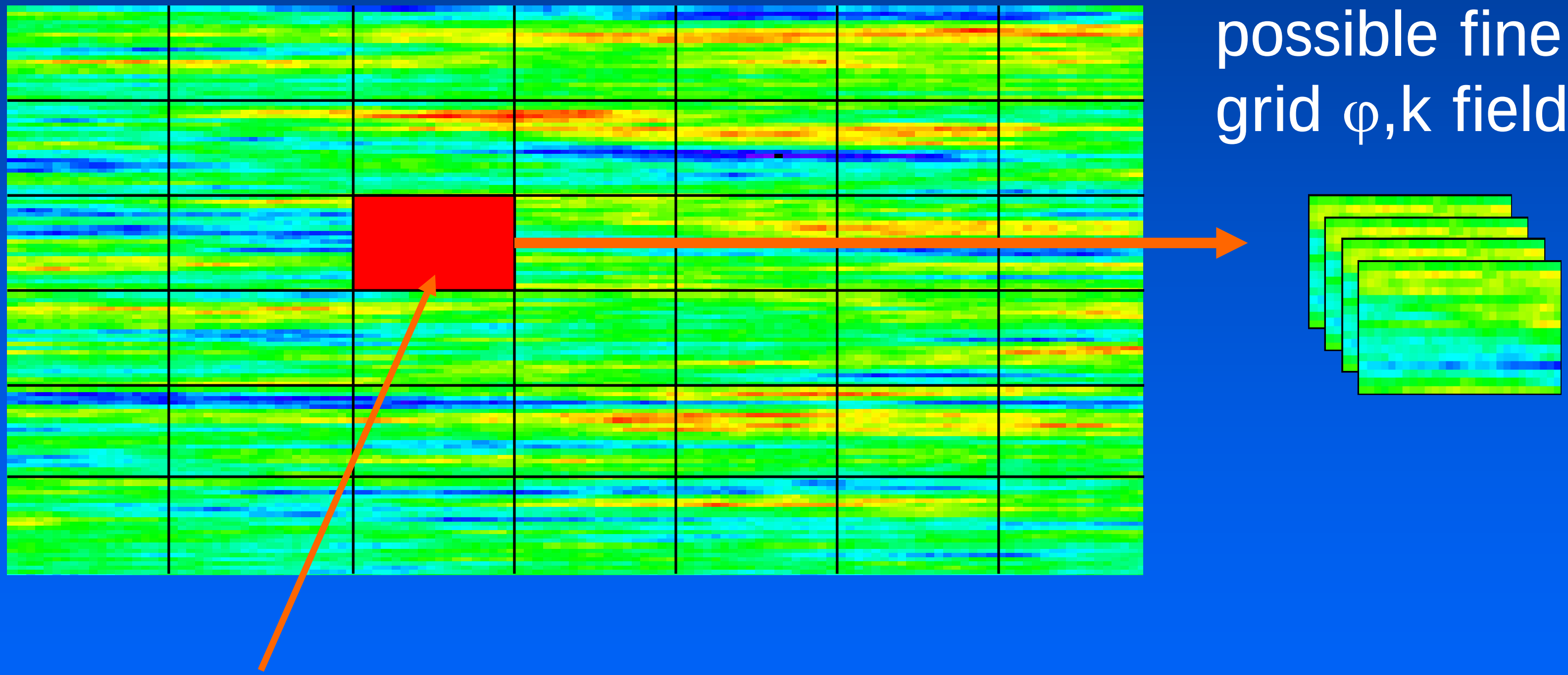
- Coarse model – fast, inaccurate
- Fine model – slow, accurate



Statistical Sub-Grid Processes

Fine grid

For a single coarse cell φ,k, many possible fine grid φ,k fields



Coarse grid cell, single φ,k

Determining Discrepancy

- Overall idea
  - Calculate discrepancy due to known ignored effects at limited set of points in parameter space
  - Compute mean discrepancy & covariance matrix
  - Interpolate/build emulator to estimate in rest of parameter space

Components of Discrepancy Model

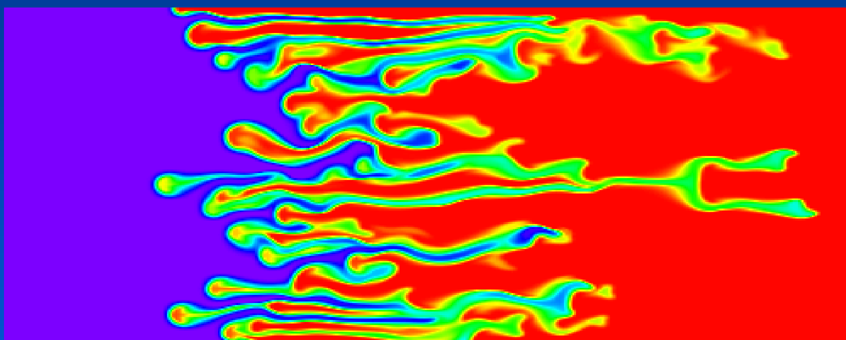
- Mean discrepancy/mean error

$$\bar{e}(t,\mathbf{m}) = \frac{1}{n}\sum_{i=1}^{n} e_i(t,\mathbf{m})$$

- Covariance Matrix

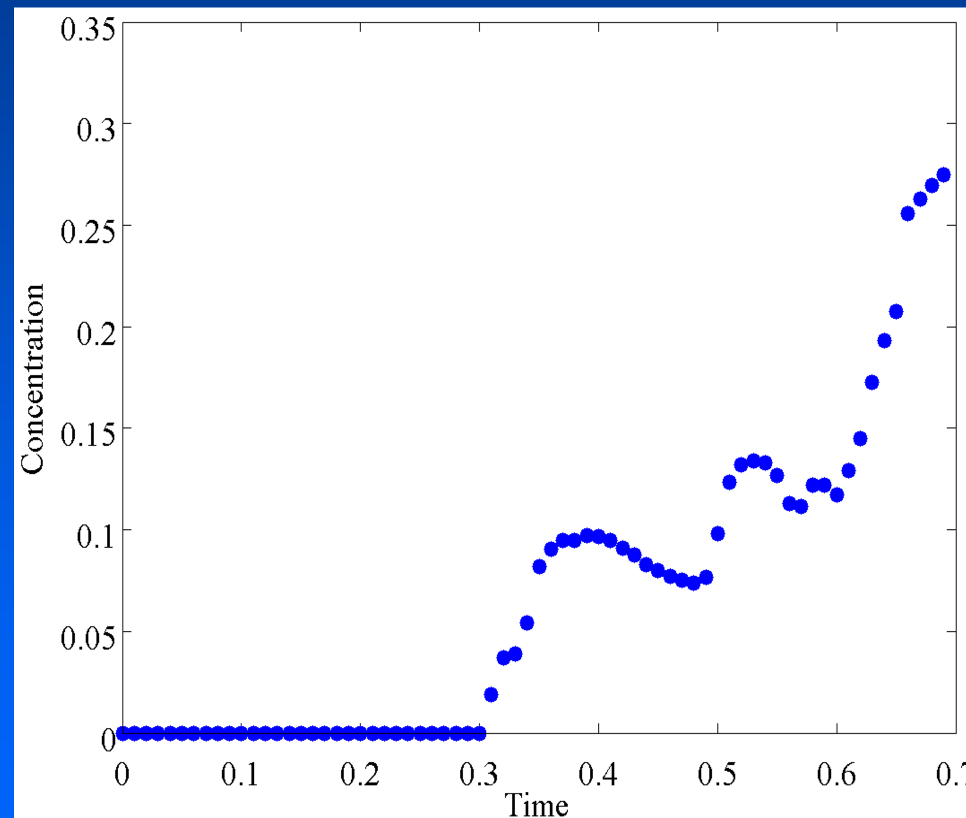
$$C(s,t) = C_d + \frac{1}{n-1}\sum_{j=1}^{n}\left(e_j(t) - \bar{e}(t)\right)\left(e_j(s) - \bar{e}(s)\right)$$

Example: Determination of Oil Viscosity

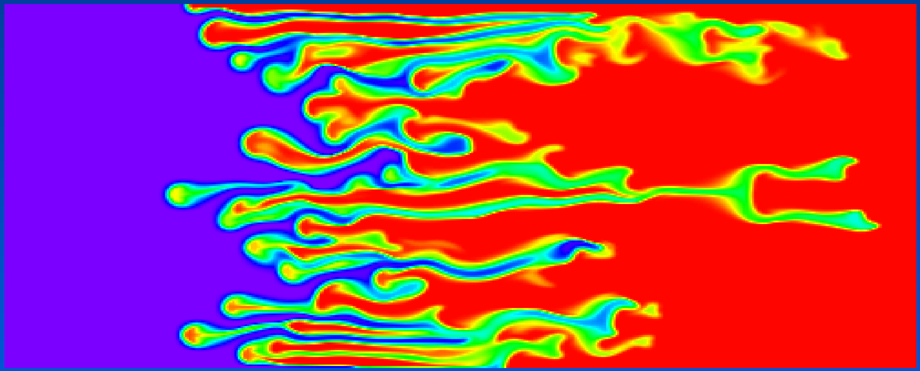


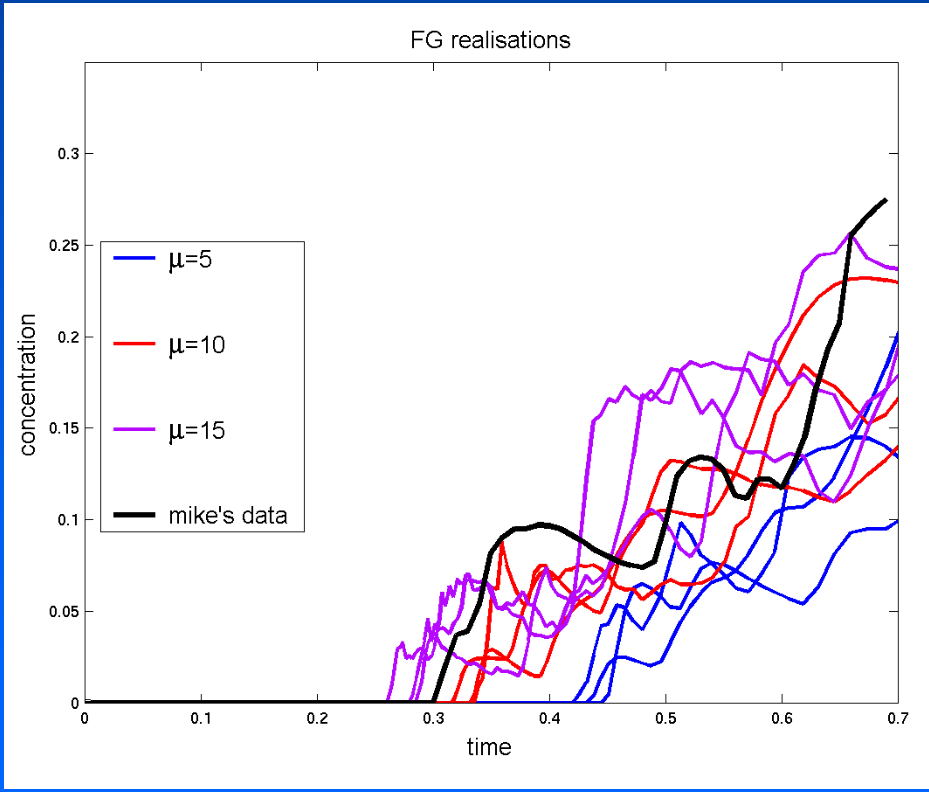
Available data:

Fluid concentration with time



Choice 1 – Fine grid Simulation

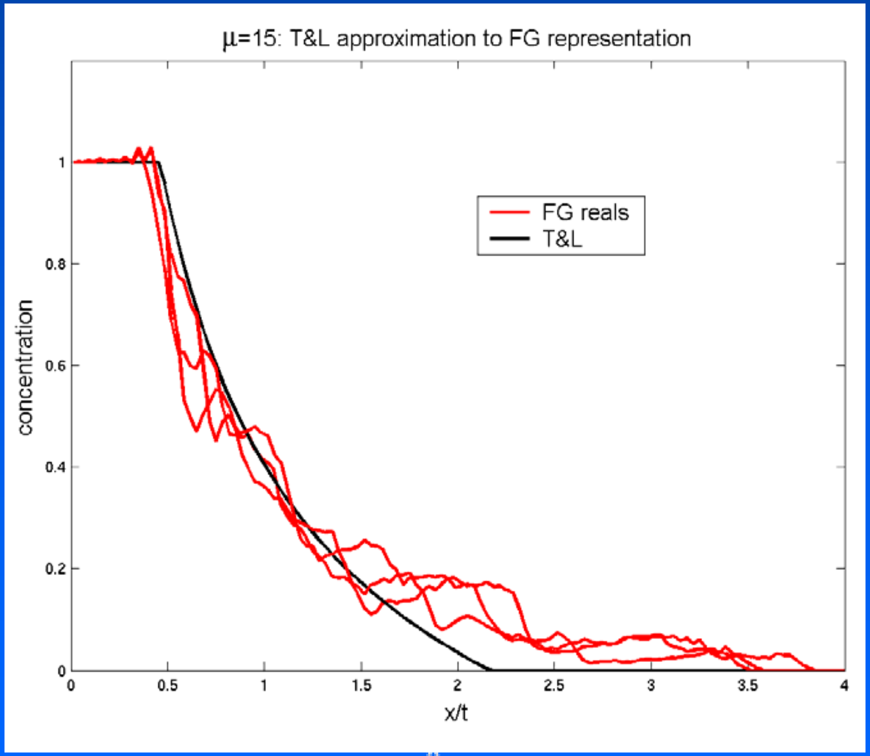


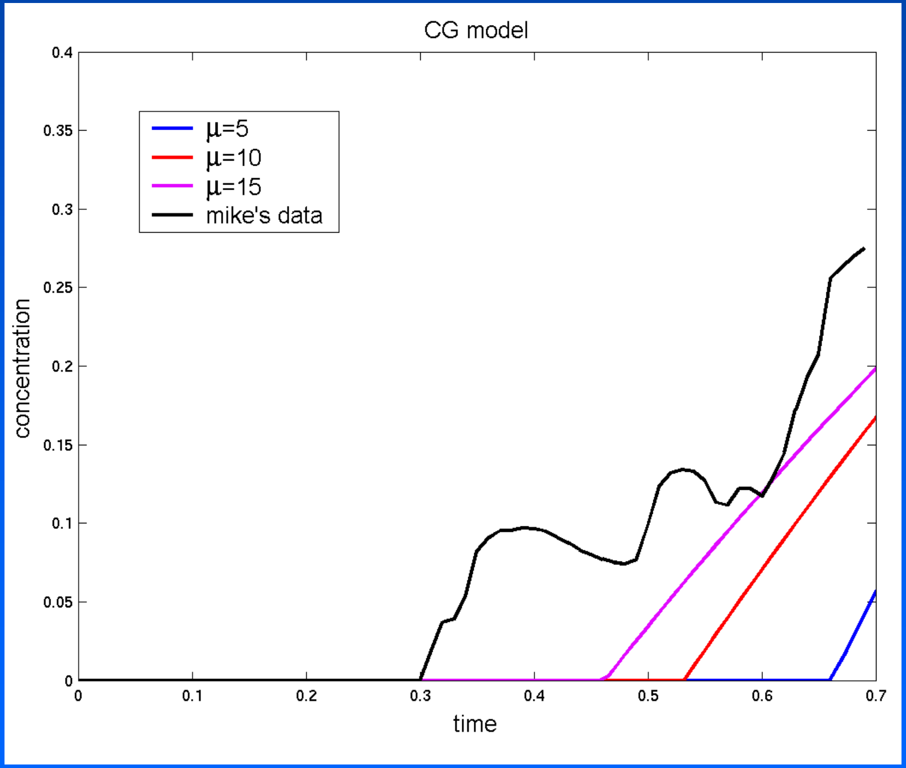


- Time consuming
- Variability in results
- Accurate

Choice 2 – Todd & Longstaff

$$f(c) = \frac{c}{c + (1-c)/M_{eff}} \qquad M_{eff} = \left(0.78 + 0.22 M^{\frac{1}{4}}\right)^4$$

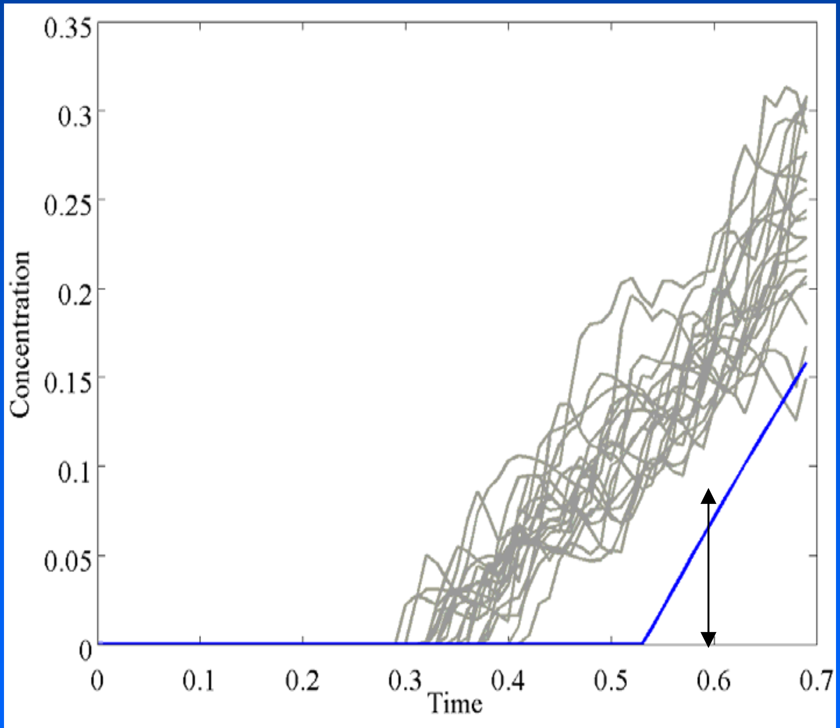


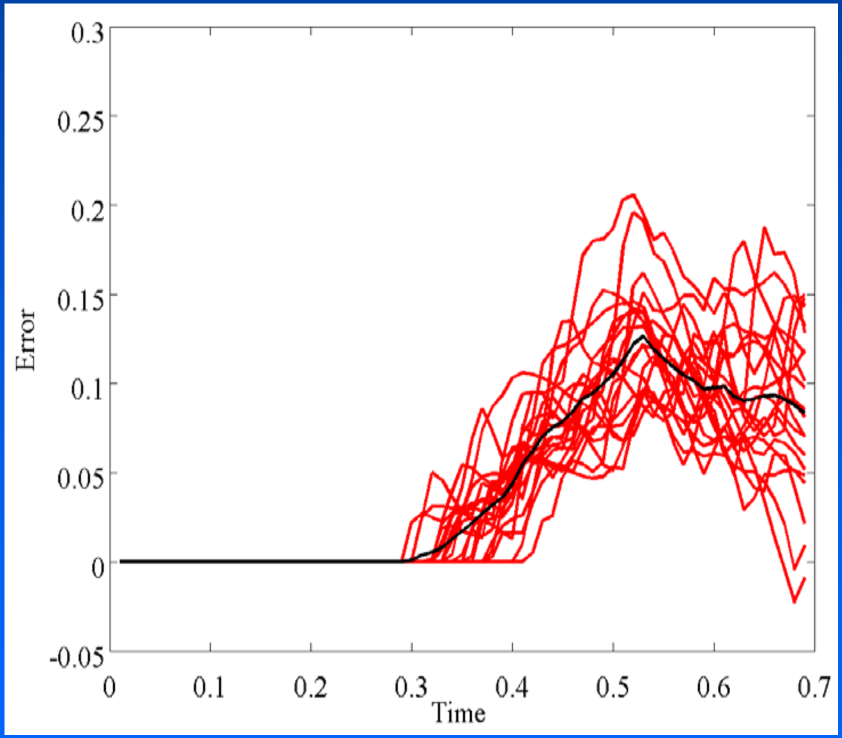


Calculate Time Varying Solution Errors

$$e_i(t,\theta) = FG_i(t,\theta) - CG(t,\theta)$$

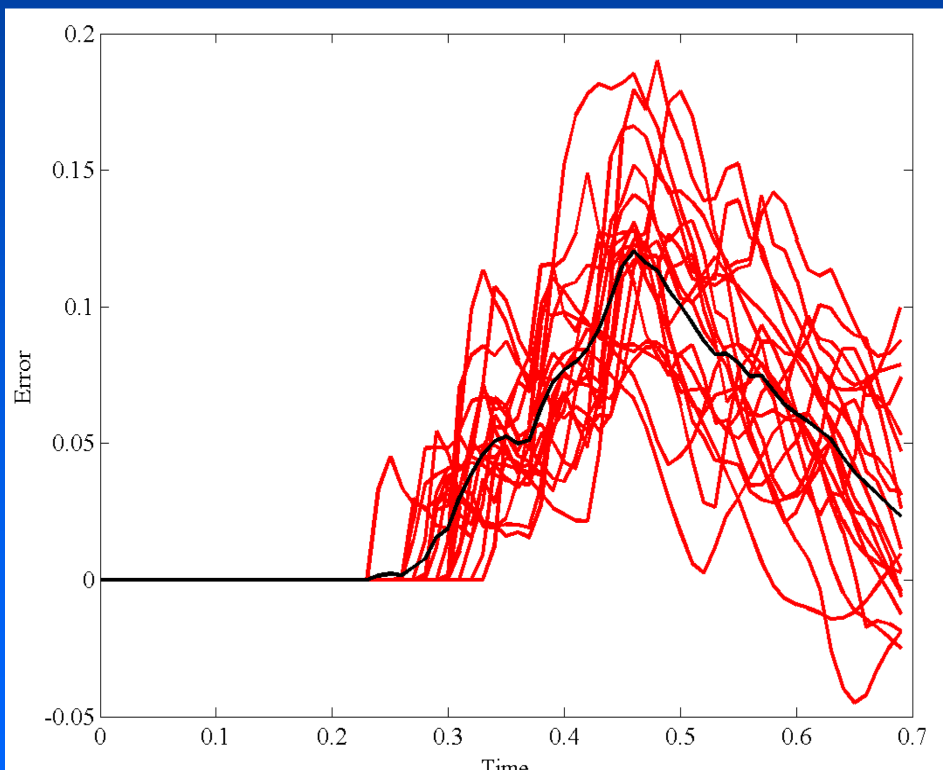
$$\overline{e}(t,\theta) = \frac{1}{n}\sum_{i=1}^{n} e_i(t,\theta)$$

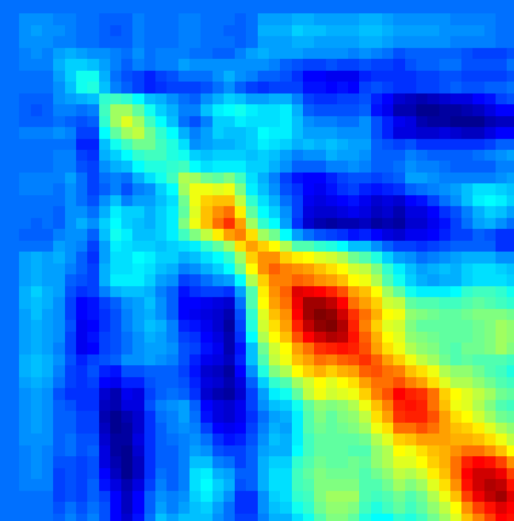




Determine Covariance Structure

$$C_{se}(s,t)=\frac{1}{n-1}\sum_{j=1}^{n}\left(e_j(t)-\overline{e}(t)\right)\left(e_j(s)-\overline{e}(s)\right)$$



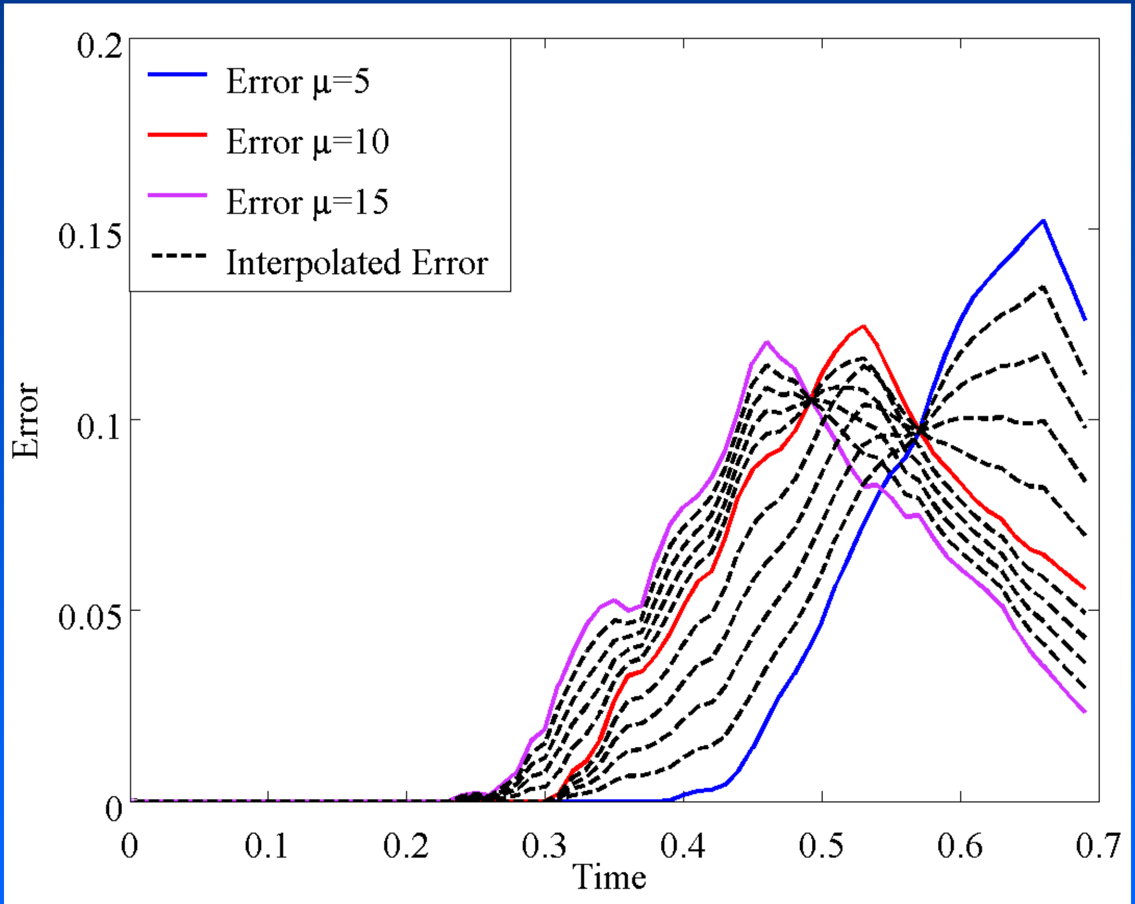


Interpolate for Mean Error and Covariance

Example:

Linear interpolation for mean error and covariance

More sophisticated scheme yields minor improvements

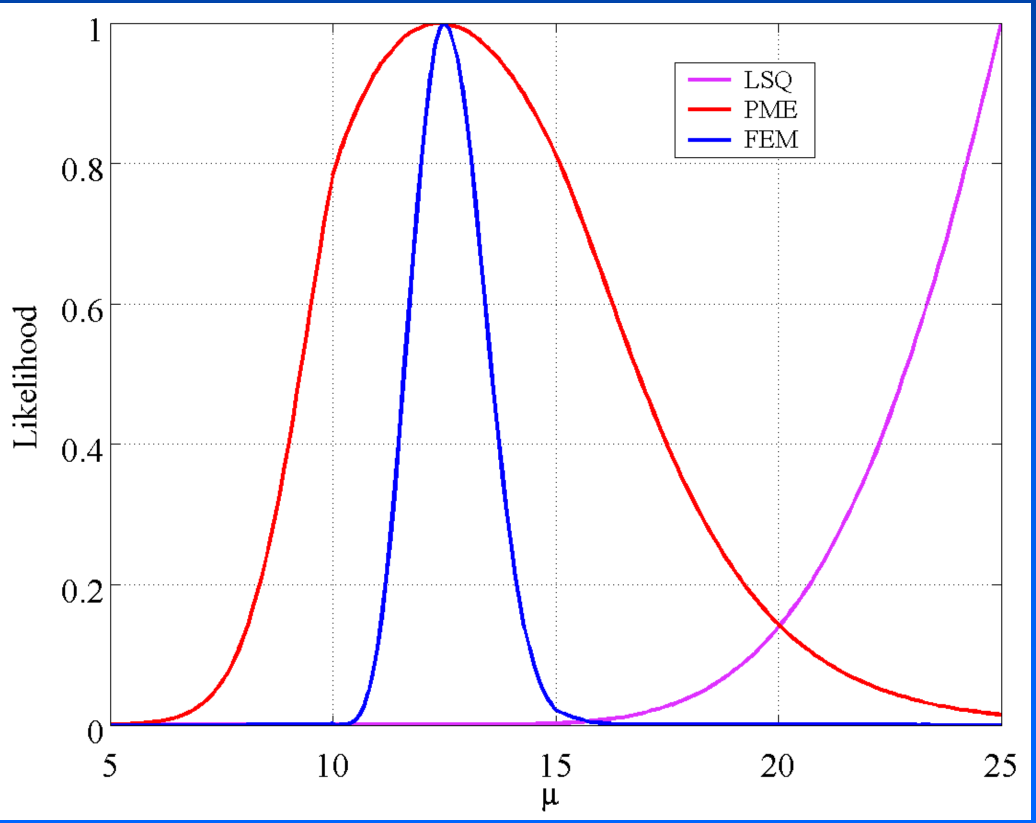


Results

True oil viscosity, μ = 13

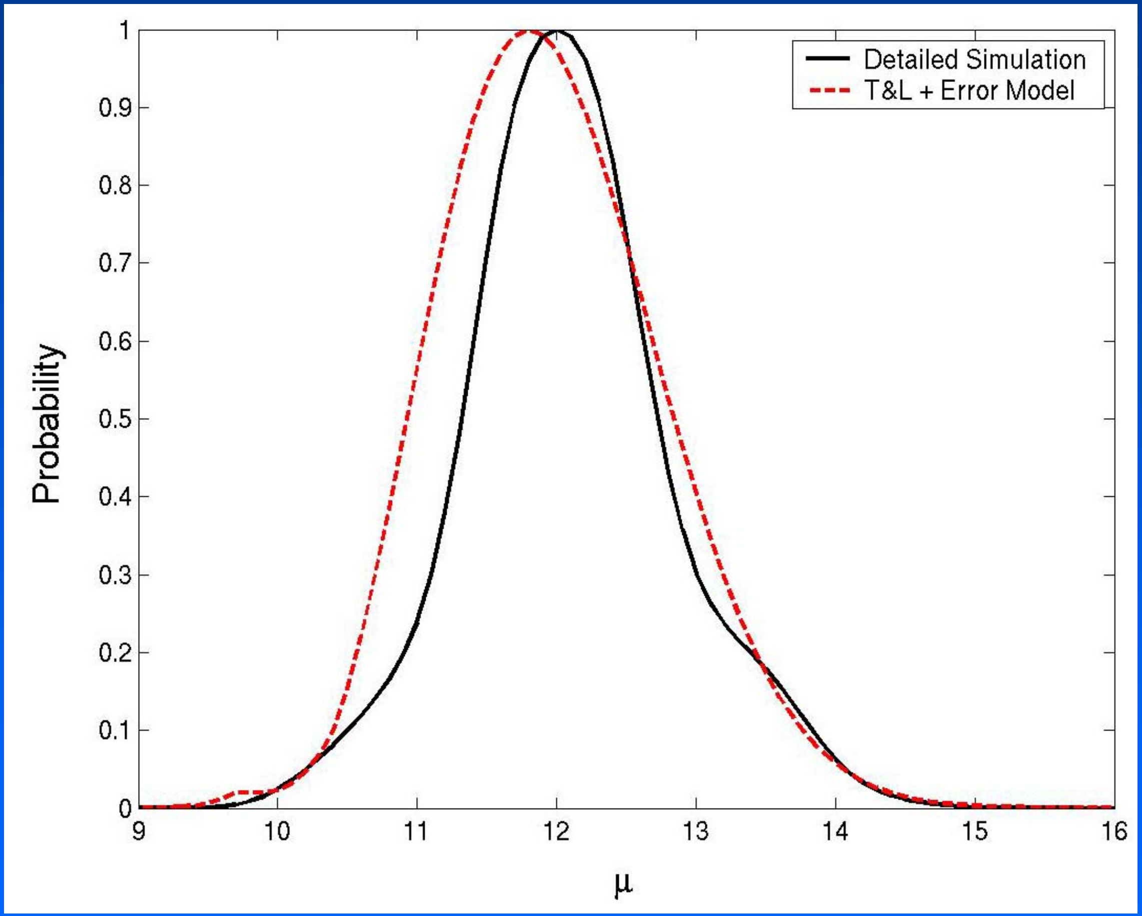
Maximum likelihood, σ=12.5

Effect of bias significantly reduced



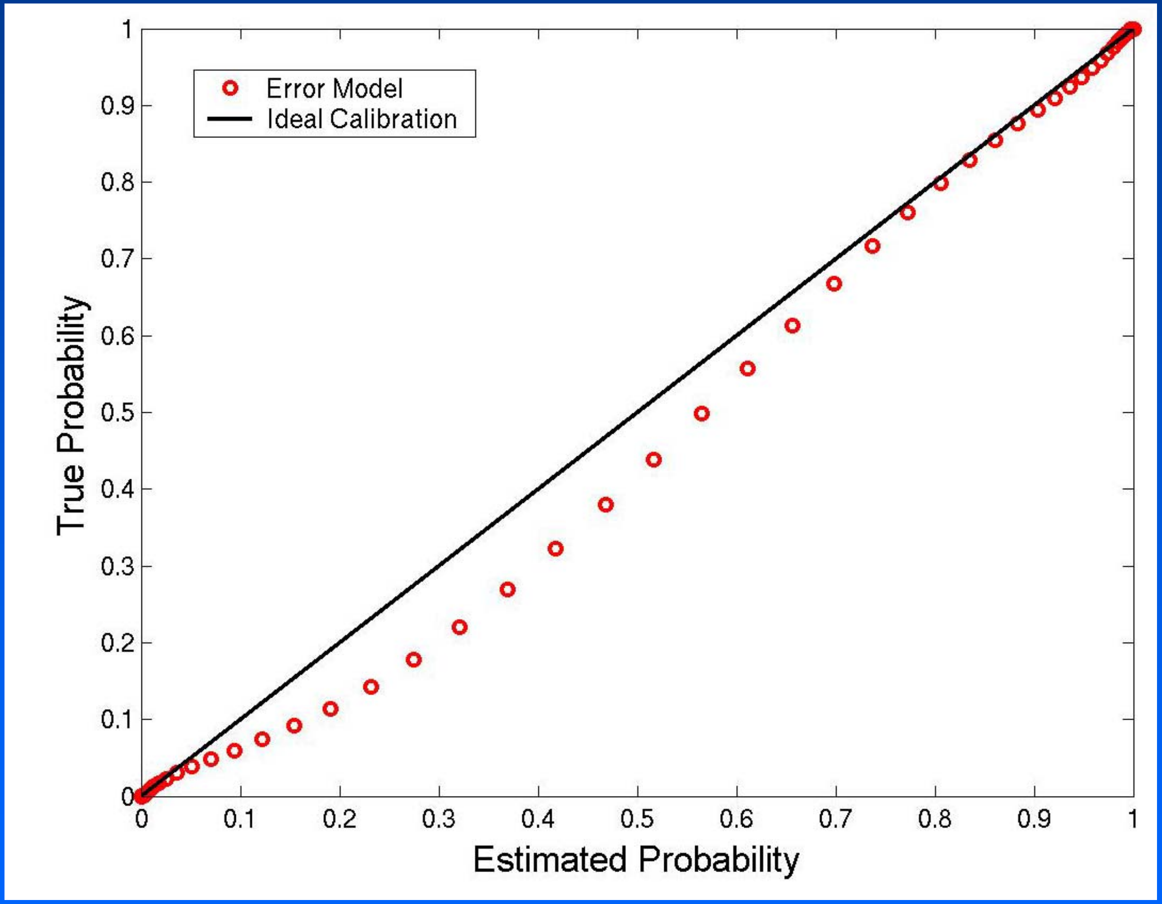
How Accurate is the Error Model?

Likelihood curves

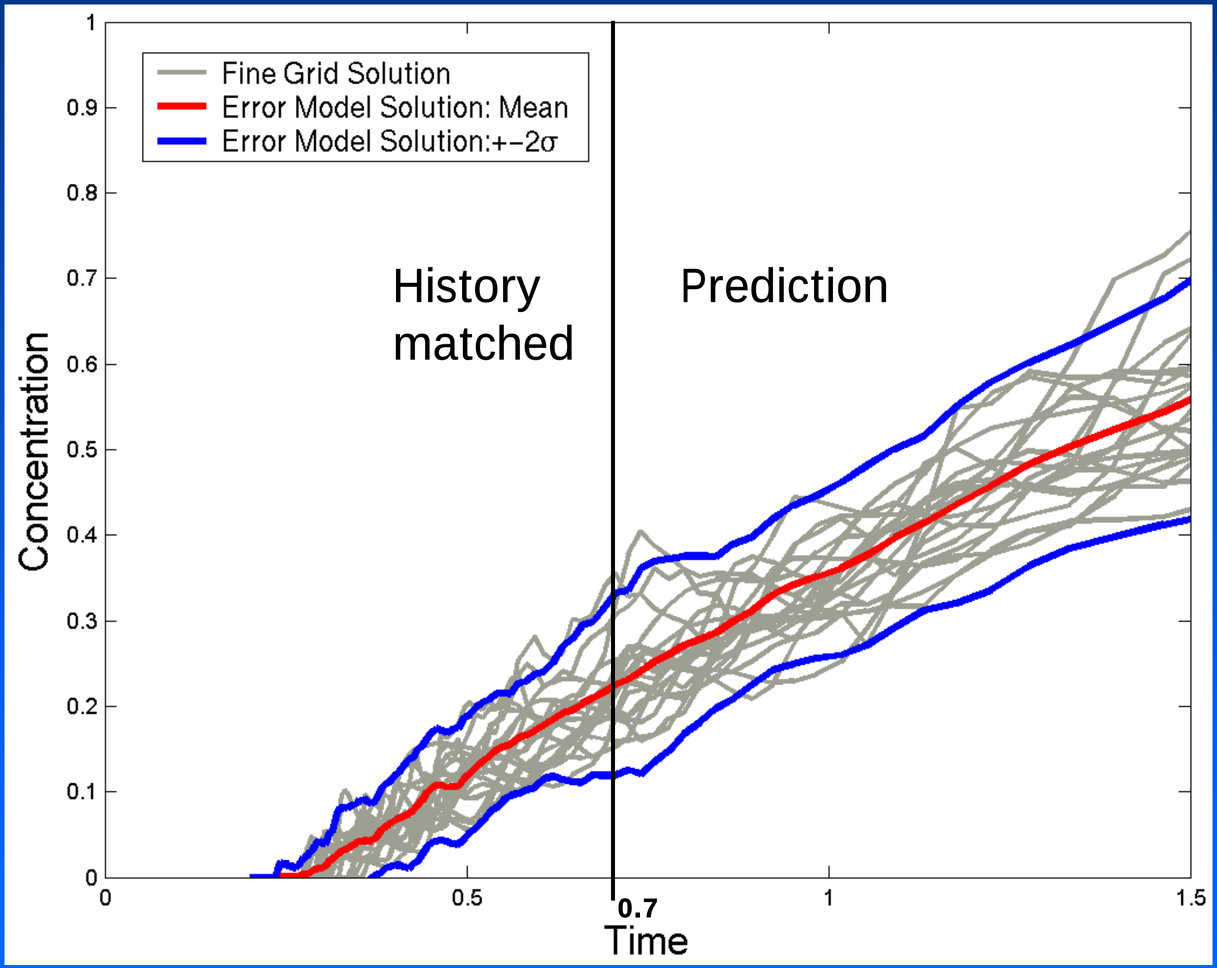


How Accurate is the Error Model?

Convert pdfs to cumulative probabilities and plot



Prediction



Fine Grid Solution
Error Model Solution: Mean
Error Model Solution:+−2σ
History matched
Prediction
Concentration
Time
0.7
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.5
1
1.5

Issues

- Estimating discrepancy model
  - Smoothing mean error/covariance data
- Emulation
  - Currently using RBF network to emulate in higher dimensional space
  - Design of locations to estimate discrepancies
- Cost
  - No point if you just effectively run fine model

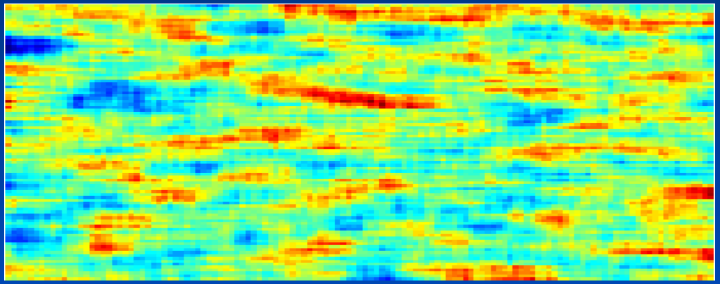
Errors for Varying Grid Sizes

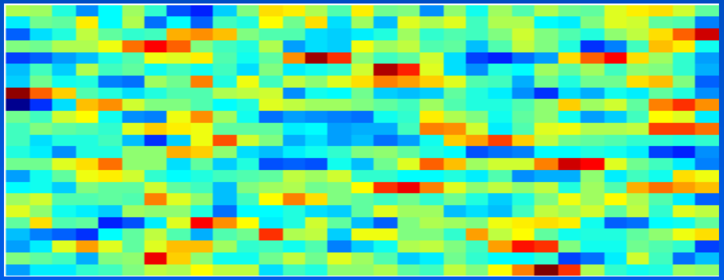
- 128x96=12288 cells (fine)

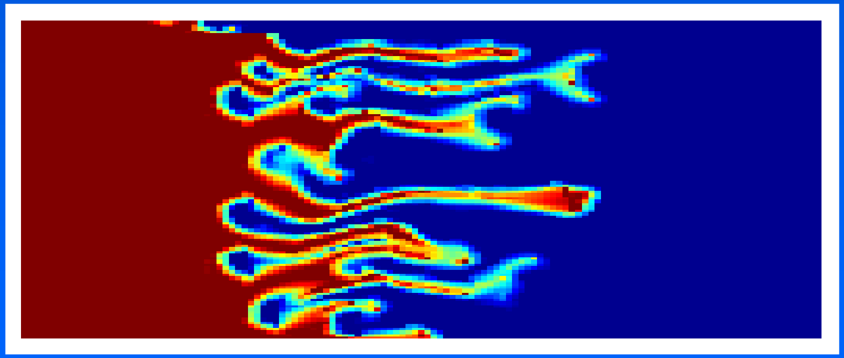
  64x48 = 3072 cells

  32x24 = 768 cells

  16x12 = 192 cells



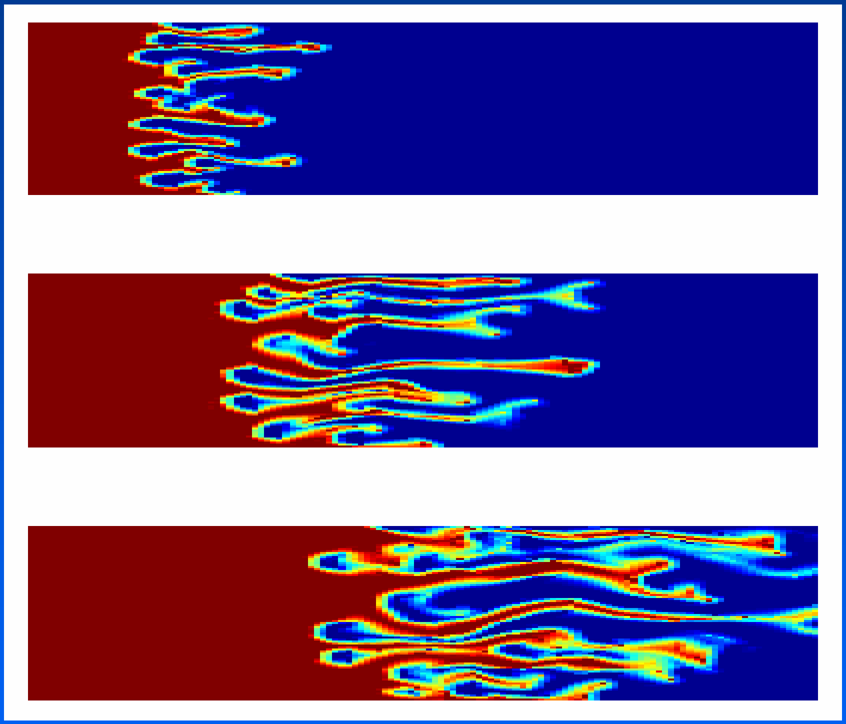


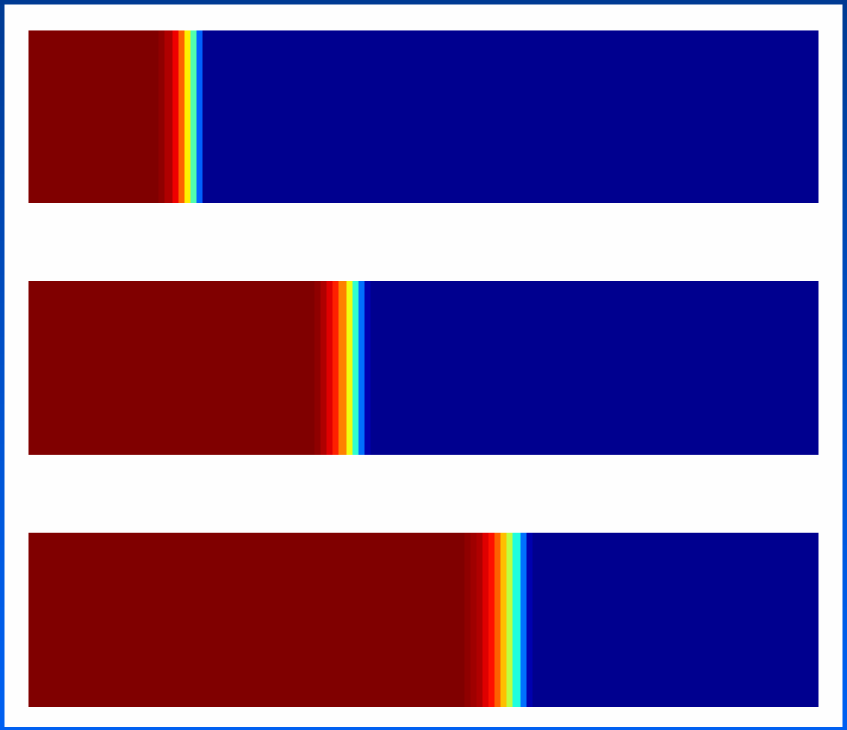


Perm:

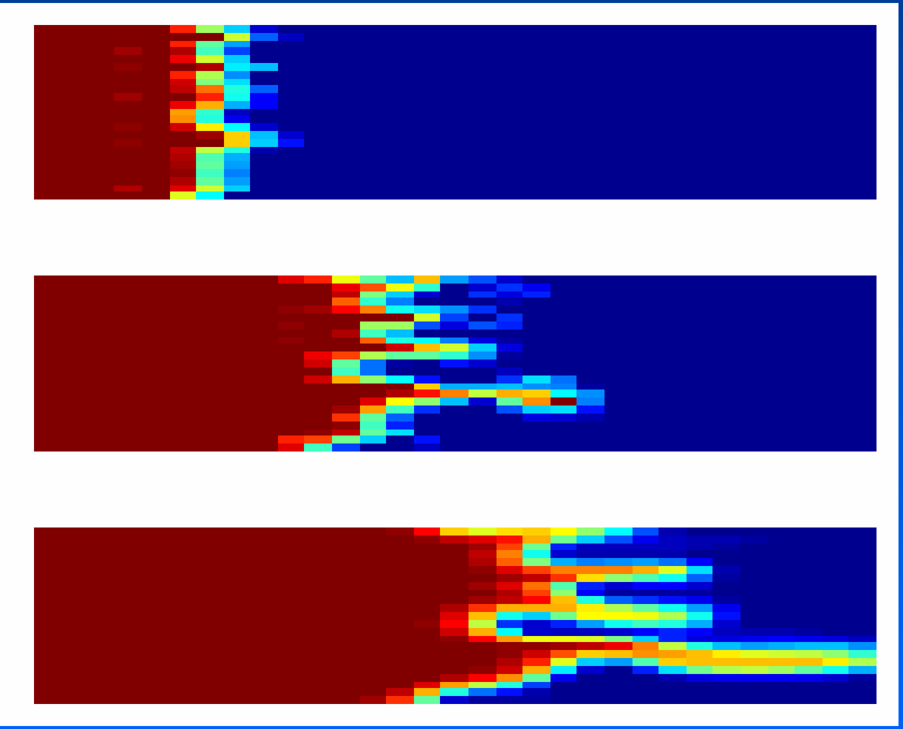
- Mean 1
- Var in log 0. 5
- Cx 0.2 Cy 0.05

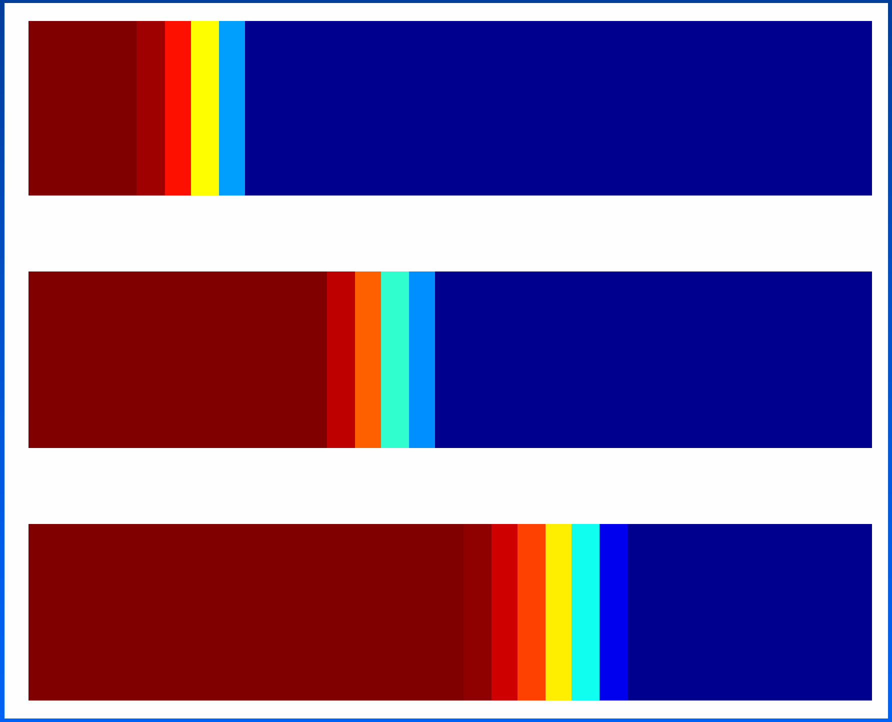
Saturation Plots: Grid 128x96



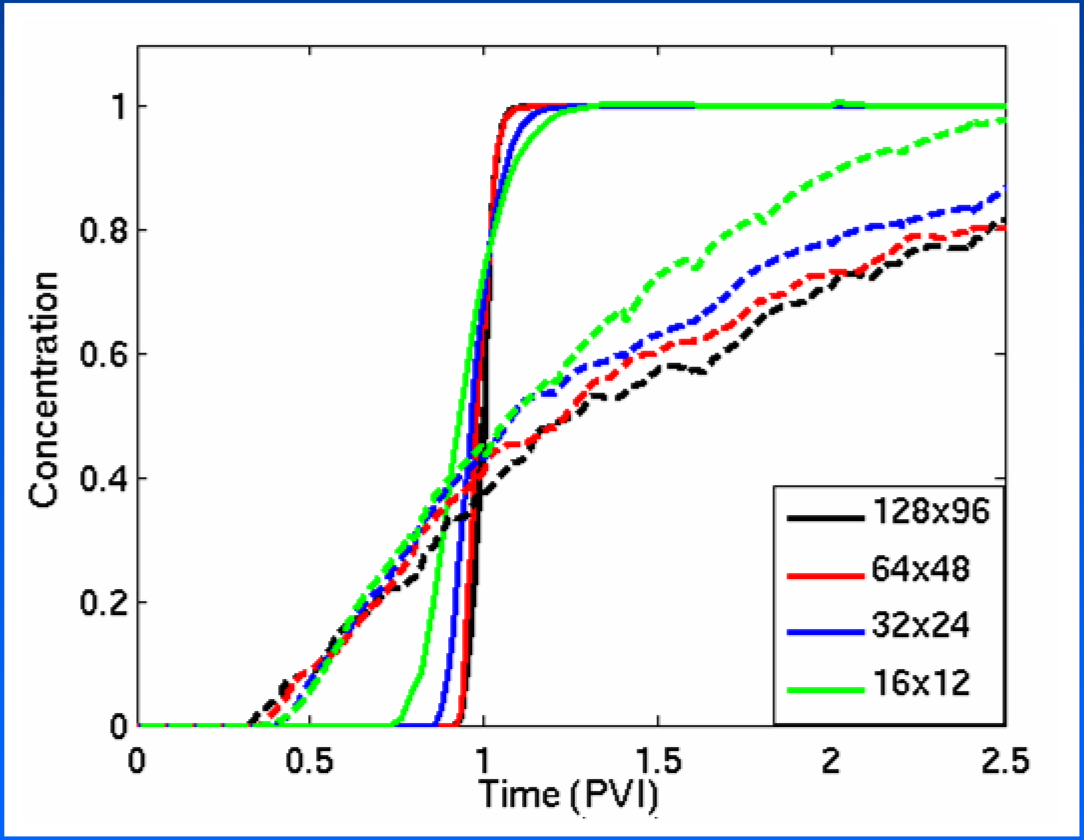


Saturation Plots: Grid 32x24



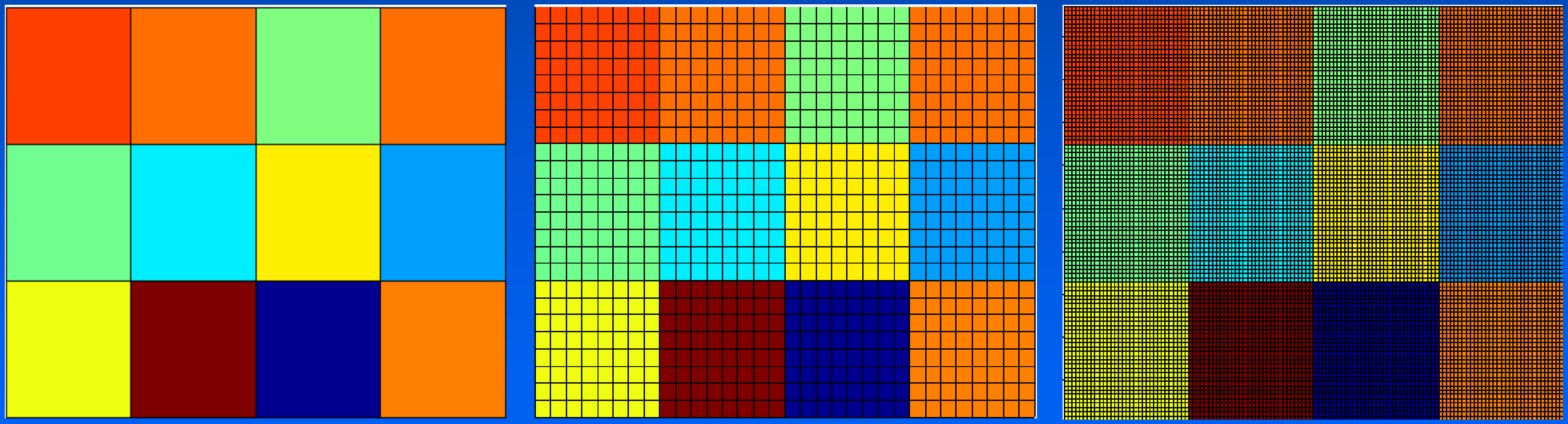


Homogeneous System



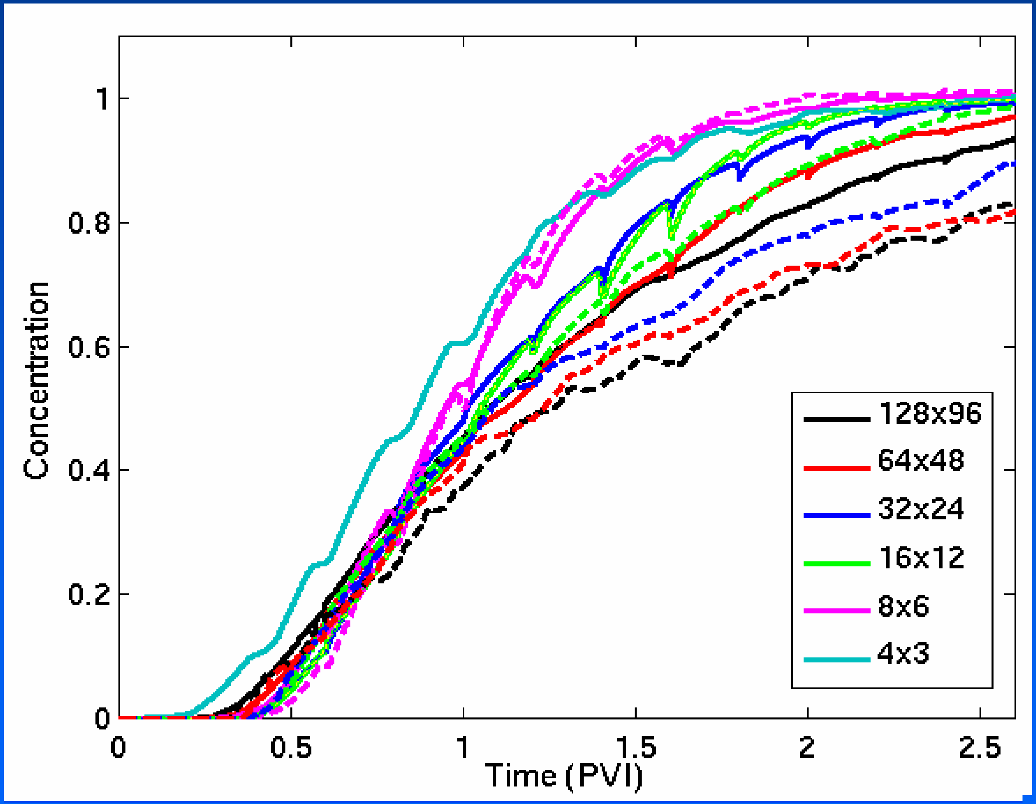
Separate the Effects

- We can separate the effects by using coarse grid data on a fine grid.



4x3 | 32x24 | 128x96

Statistically Equivalent c.f. Overlaying grids



Comparing statistically equivalent data for a range of resolutions (solid) with coarse grid data on finer resolution grids (dotted)

Perm:

- Mean: 1
- Var in log 0.5
- Cx 0.2 Cy 0.05

Model Errors

- Kennedy & O'Hagan (2001) approach
  - Determine model errors as part of inference
  - Important to account for model errors
  - Assume model errors independent of parameters

Kennedy and O’Hagan

$$z_i = \boxed{\rho\eta(x_i, \theta) + \delta(x_i)} + e_i$$

z_i observations

ρ scalar

η simulator

x_i variable input

θ history match input

δ model inadequacy

e_i observation error

$$\varsigma(x_i) = \rho\eta(x_i, \theta) + \delta(x_i)$$

truth

Relationship to Kennedy & O'Hagan

- Kennedy & O'Hagan approach

$$z(t) = \rho\eta(t,\theta) + \delta(t) + \varepsilon$$

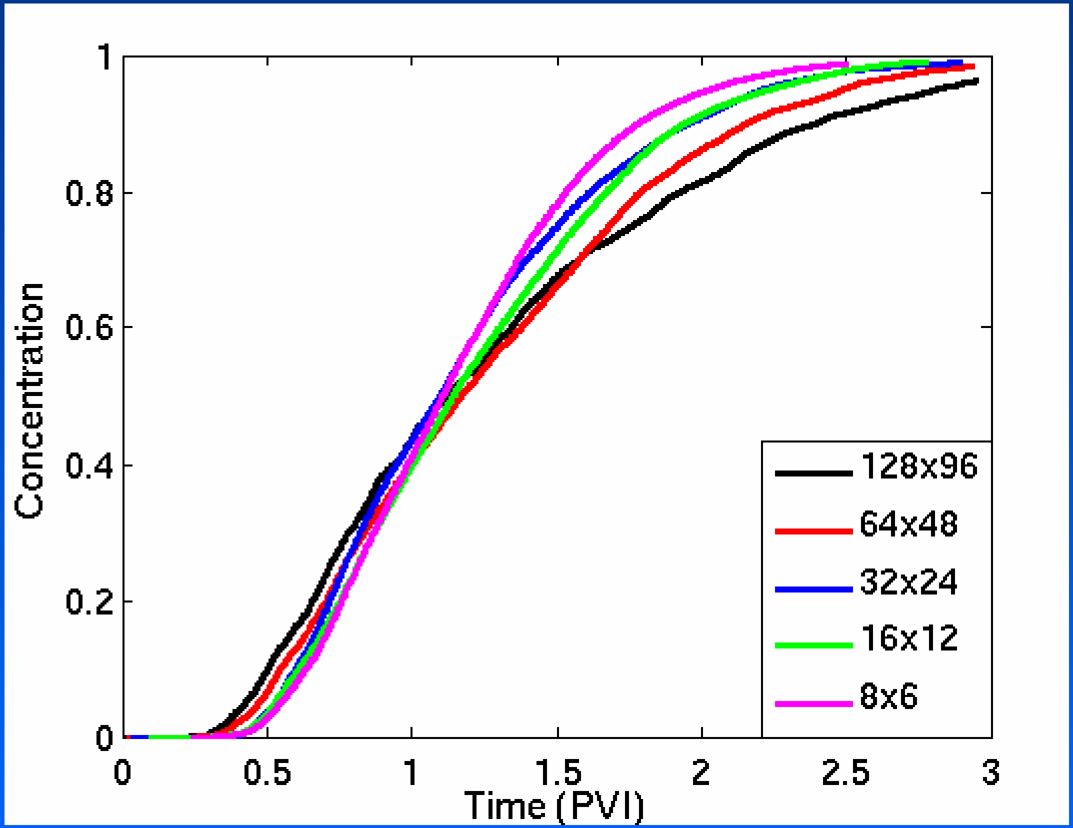
$$e(t) = (\rho - 1)\eta(t,\theta) + \delta(t) + \varepsilon$$

- Our approach

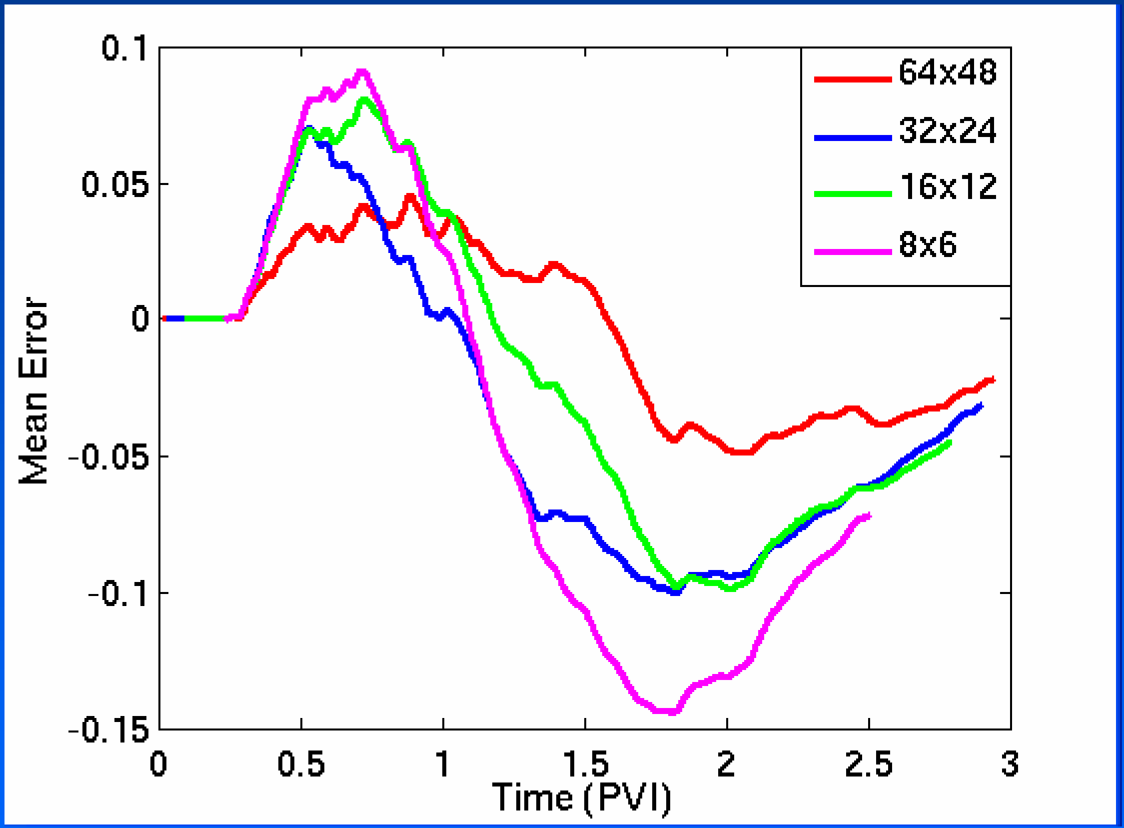
$$z(t) = \eta(t,\theta) + \delta(t,\theta) + \varepsilon$$

$$e(t) = \delta(t,\theta) + \varepsilon$$

Solution as a function of grid size

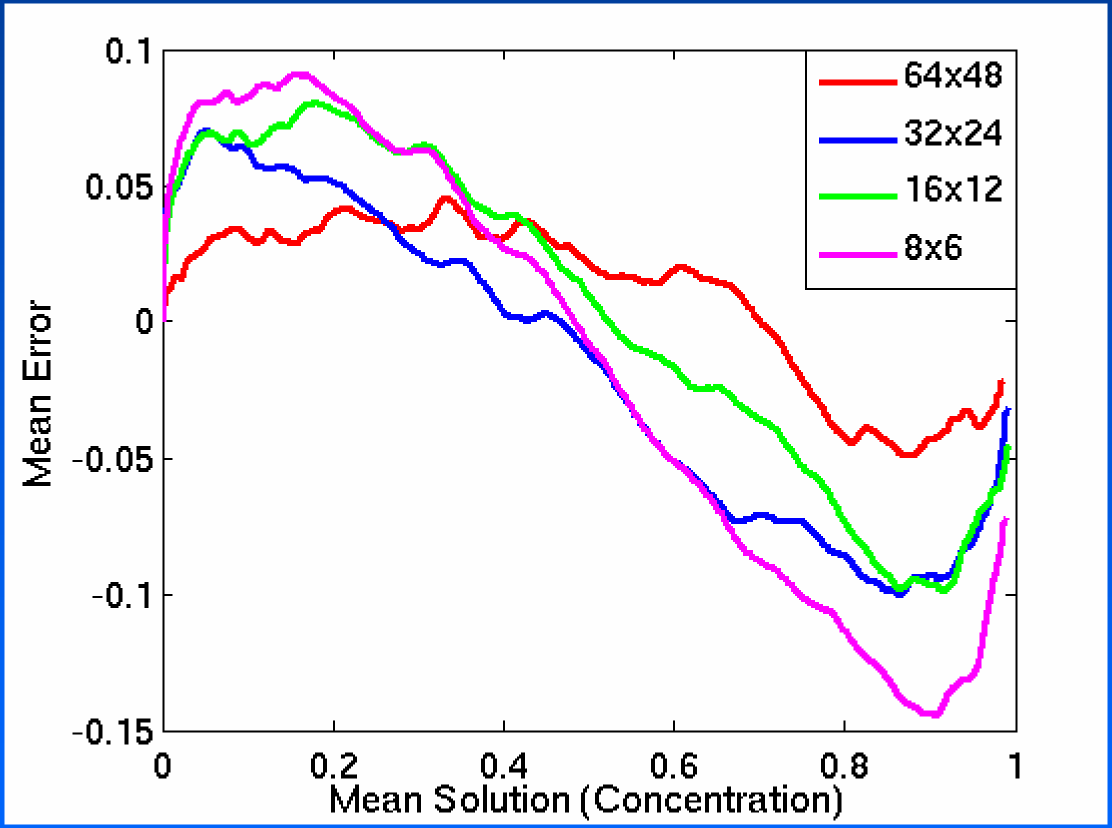


Error as a function of grid size



Error as a function of Solution

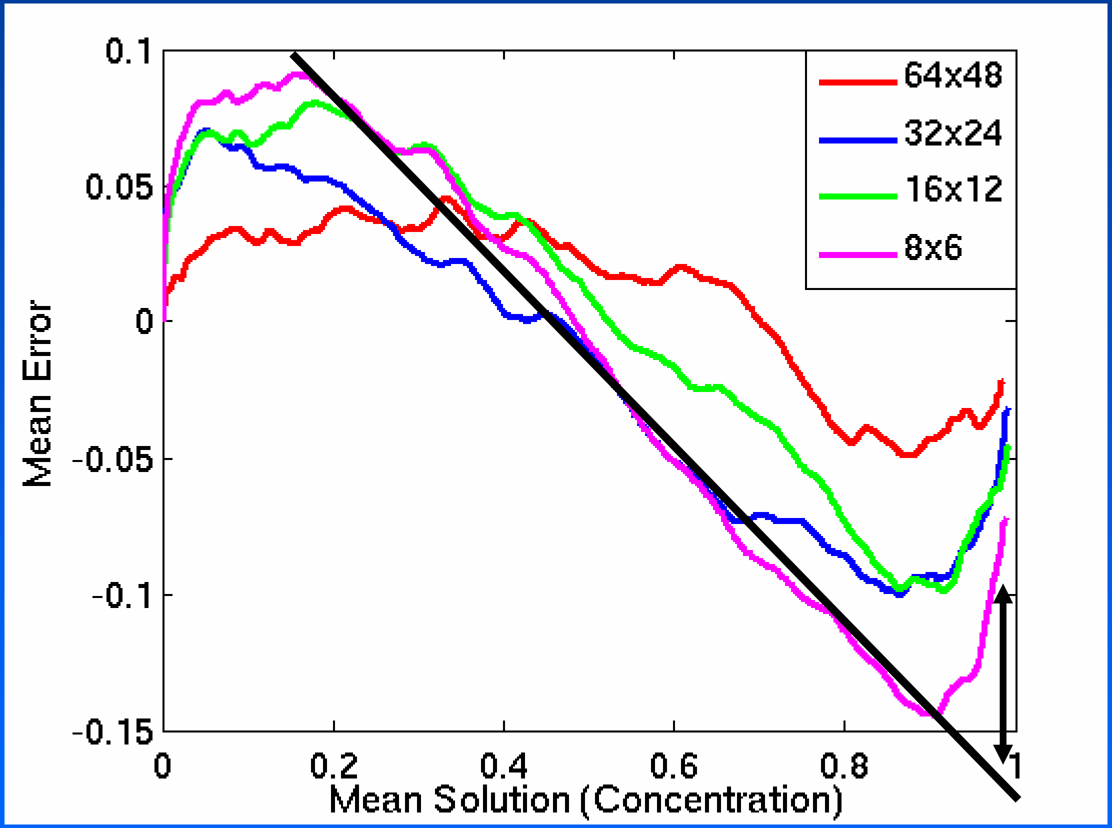
$$e(x_i, \theta) = (\rho - 1)\eta(x_i, \theta) + \delta(x_i)$$



Uncertainty Quantification

Error as a function of Solution

$$e(x_i,\theta) = (\rho - 1)\eta(x_i,\theta) + \delta(x_i)$$



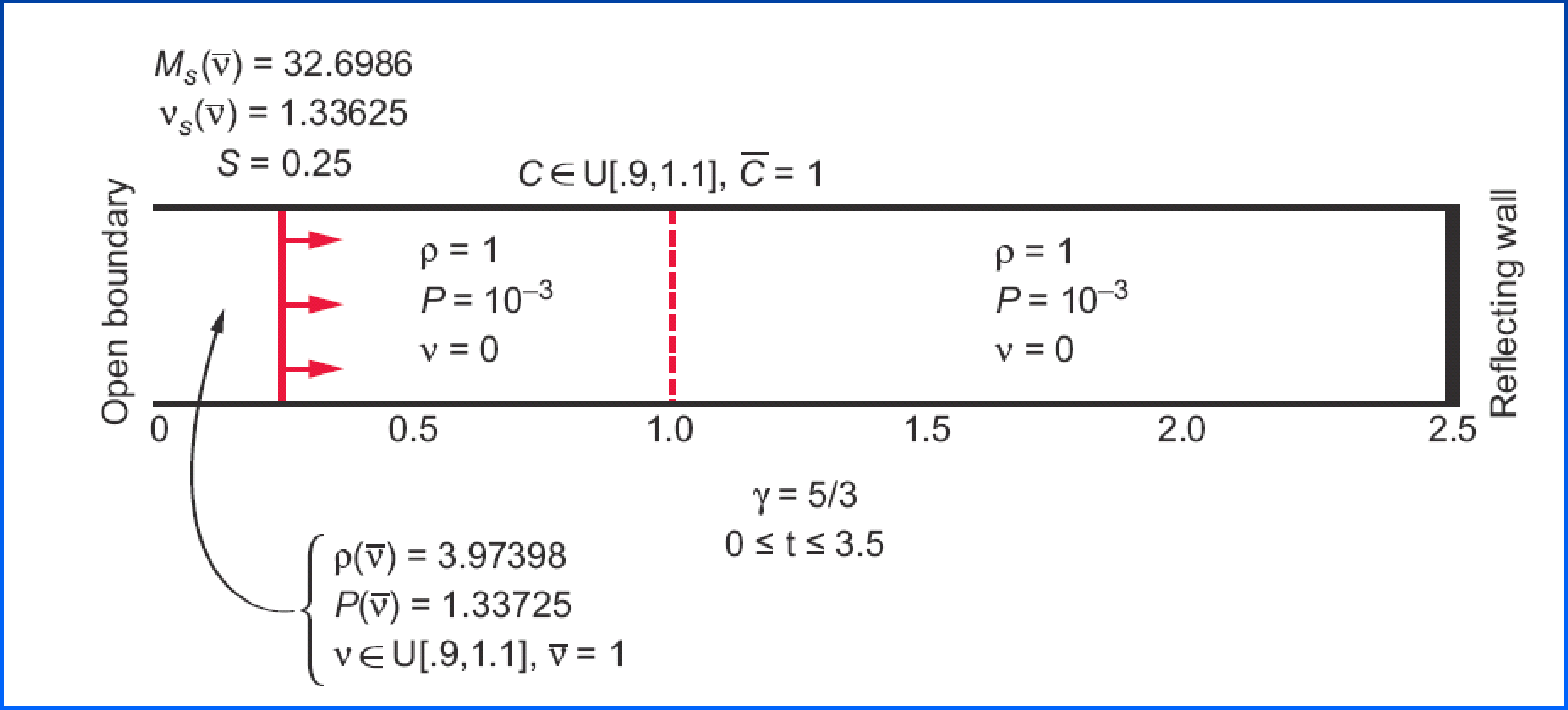
ρ = 0.68

$\delta(x)$

Gas Dynamics Example

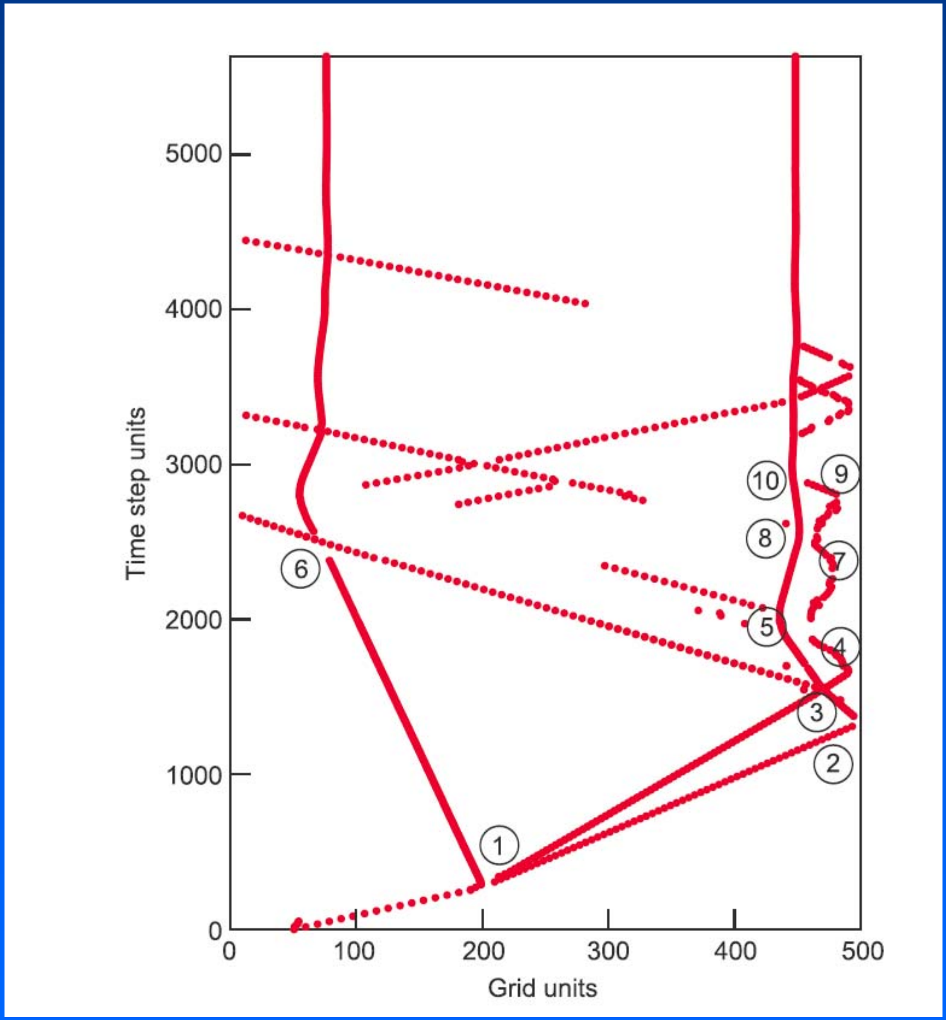
- Additional description in Los Alamos Science
  - Volume 29
  - www.lanl.gov/science
  - And references therein
- Similar approach to porous media example
  - Addresses uncertainties in shock strength/timing
  - Builds complex shock-interaction model from single shock-contact study

Geometry of Shock Tube



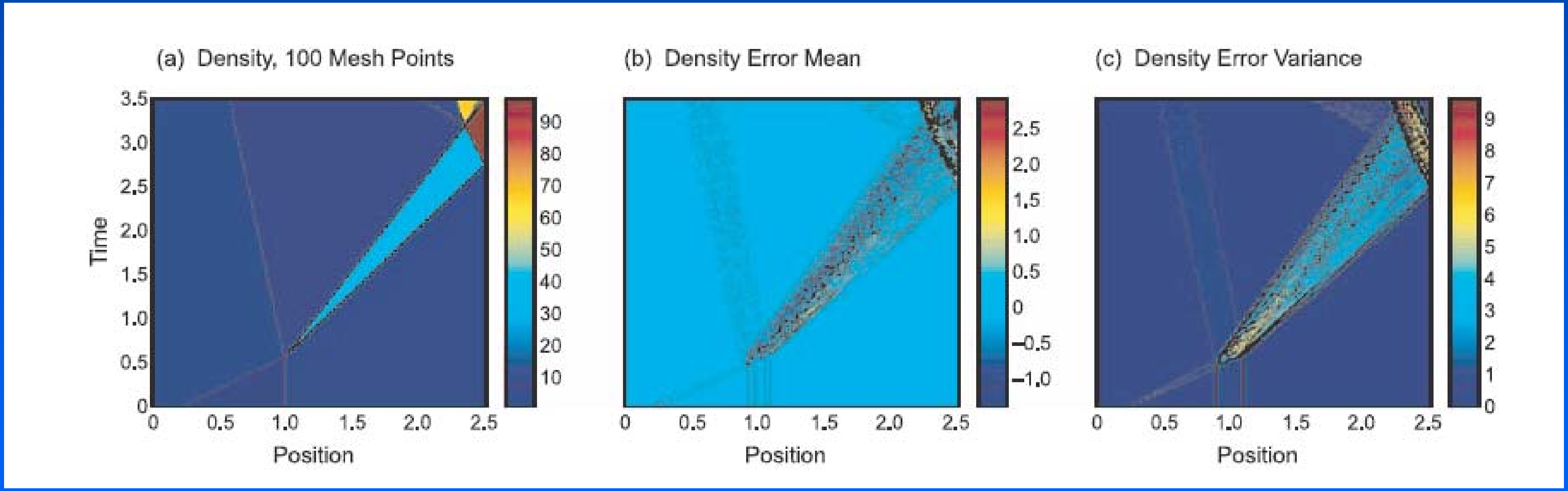
Dynamics of Flow

- Reverberating shock



Computed Errors

- Composition law for errors
  - All from original study of single shock-contact



Glimm, J., J. W. Grove, Y. Kang, T. W. Lee, X. Li, D. H. Sharp, et al. *2003.* "Statistical Riemann Problems and a Composition Law for Errors in Numerical Solutions of Shock Physics Problems," Los Alamos National Laboratory document LA-UR-03-2921. *SIAM J. Sci. Comput.* (in press).

Summary

- Solution Error Model
  - Compute discrepancy for known additional phenomena
  - Interpolate over parameter space
  - Mean error non-zero and function of parameters
- Kennedy & O’Hagan
  - Similar in spirit
  - Model inadequacy determined from data
- Applications
  - Gas injection problem
  - Gas dynamics